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$E = 27^\circ\ 15' + 46^\circ = 73^\circ\ 15'$. Since $10n = 46$, I employ in the second case the equation (2) as follows:—

$\log 10n$	1·65321	1·66276	1·67210
$\log \frac{(1^\circ)''}{R''}$	8·24188	8·24188	8·24188
	9·89509	9·90464	9·91398
$\log e$	9·92315	9·92315	9·92315
$\log \sin (M + 10n \, . \, 1^\circ)$	9·97882	9·98117	9·98338
	9·90197	9·90432	9·90653

Here $10n = 45 + \frac{688}{688+32} = 45·956$; but $45^\circ·956 = 45^\circ\ 57'\ 22''$. Therefore $E = 27^\circ\ 15'\ 7''·4 + 45^\circ\ 57'\ 21''·6$; or, $E = 73^\circ\ 12'\ 29''$, a value exact to within a few seconds.

The definitive correction ΔE, may be obtained from

$$\Delta E = \frac{M + e \sin E - E}{1 - e \cos E}.$$

Naples, Feb. 7, 1877.

Note on a Transformation of Lagrange's Equations of Motion in Generalised Coordinates, which is convenient in Physical Astronomy. By Robert S. Ball, LL.D., F.R.S., Royal Astronomer of Ireland.

I can hardly suppose that the following Note contains anything which has not already been published; but as I have not met with this transformation in my reading, nor conversed with anyone who is acquainted with it, I am led to believe that, even if it be not new, it is certainly not so well known as its convenience deserves.

Lagrange's Equations of Motion in Generalised Coordinates are not, in their usual form, particularly well adapted for Physical Astronomy. In a problem in Physical Astronomy, what we want to learn is the *relative* coordinates of a system, while its *absolute* situation in space does not concern us. Lagrange's equations, however, involve *both* the relative and the absolute coordinates, and we are obliged to remove the latter by elimination. It is the object of this Note to point out a very simple transformation of Lagrange's equations which obviates the inconvenience of having to perform the elimination. This advantage is gained without detracting from the simplicity and directness which render Lagrange's equations such exquisite instruments of analysis.

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If q be one of the generalised coordinates, and T and V be the kinetic and potential energies of the system, then Lagrange's equations have the form

$$\frac{d}{dt}\left(\frac{dT}{d\dot{q}}\right)-\frac{dT}{dq}=-\frac{dV}{dq};$$

V depends solely upon the relative coordinates of the system. It is therefore only the function T which requires to be purged from the three objectionable quantities. This is effected by simply replacing T by a different function, which we shall denote by T′.

We proceed to show that this substitution is allowable, and that it answers the end for which it is intended.

Let p_1, p_2, p_3 be the rectangular coordinates of the centre of inertia P of one of the bodies with reference to axes fixed in space. Let the relative generalised coordinates of the system be $q_1, \ldots q_{n-3}$. When the last-mentioned coordinates and the corresponding generalised velocities are known, then everything is known about the system except its actual situation in space. What we want to obtain is a system of $n-3$ equations in $q_1, \ldots q_{n-3}$, from which p_1, p_2, p_3 are absent.

Let x_1, y_1, z_1 be the rectangular coordinates of the centre of inertia of one of the masses m_1 with respect to three rectangular axes through P parallel to the fixed axes, then we have

$$T=\frac{1}{2}\Sigma m_1(\dot{p}_1+\dot{x}_1)^2+\frac{1}{2}\Sigma m_1(\dot{p}_2+\dot{y}_1)^2+\frac{1}{2}\Sigma m_1(\dot{p}_3+\dot{z}_1)^2+R_o,$$

when R_o is that function of $q_1, \ldots q_{n-3}$ and $\dot{q}_1, \ldots \dot{q}_{n-3}$ which expresses the kinetic energy of the rotations of the bodies of the system about their centres of inertia.

Then we have

$$T=\frac{1}{2}(\dot{p}_1^2+\dot{p}_2^2+\dot{p}_3^2)\Sigma m_1+\dot{p}_1\Sigma m_1\dot{x}_1+\dot{p}_2\Sigma m_1\dot{y}_1+\dot{p}_3\Sigma m_1\dot{z}_1$$
$$+\frac{1}{2}(\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1^2)\Sigma m_1+R_o.$$

We first form the generalised equations of motion in Lagrange's form corresponding to the coordinate p_1:

$$\frac{dT}{d\dot{p}_1}=\dot{p}_1\Sigma m_1+\Sigma m_1\dot{x}_1,$$

$$\frac{dT}{dp_1}=0,$$

$$\frac{dV}{dp_1}=0;$$

whence the equation of motion is

$$\frac{d}{dt}(\dot{p}_1\Sigma m_1+\Sigma m_1\dot{x}_1)=0,$$

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and two similar equations, corresponding to p_2 and p_3. If then u, v, w be constants with respect to the time,

$$\dot{p}_1 \Sigma m_1 + \Sigma m_1 \dot{x}_1 = u \Sigma m_1,$$

$$\dot{p}_2 \Sigma m_1 + \Sigma m_1 \dot{y}_1 = v \Sigma m_1,$$

$$\dot{p}_3 \Sigma m_1 + \Sigma m_1 \dot{z}_1 = w \Sigma m_1.$$

We now proceed to form the equations in $q_1, \ldots q_{n-3}$. We shall use the letter q as typical of the series.

If we make

$$\text{T}' = \frac{1}{2}(\dot{x}_1{}^2 + \dot{y}_1{}^2 + \dot{z}_1{}^2)\,\Sigma m_1$$

$$-\frac{1}{2\Sigma m_1}\{(\Sigma m_1 \dot{x}_1)^2 + (\Sigma m_1 \dot{y}_1)^2 + (\Sigma m_1 \dot{z}_1)^2\} + \text{R}_o,$$

then it is easily verified that

$$\text{T} = \frac{1}{2}(u^2 + v^2 + w^2)\,\Sigma m_1 + \text{T}'.$$

Making this substitution in the equation of motion corresponding to q, viz.

$$\frac{d}{dt}\left(\frac{d\text{T}}{d\dot{q}}\right) - \frac{d\text{T}}{dq} = -\frac{d\text{V}}{dq},$$

the equation becomes

$$\left\{\frac{d}{dt}\left(\frac{du}{d\dot{q}}\right) - \frac{du}{dq}\right\} u\Sigma m' + \left\{\frac{d}{dt}\left(\frac{dv}{d\dot{q}}\right) - \frac{dv}{dq}\right\} v\Sigma m' + \left\{\frac{d}{dt}\left(\frac{dw}{d\dot{q}}\right) - \frac{dw}{dq}\right\} w\Sigma m'$$

$$+\frac{d}{dt}\left(\frac{d\text{T}'}{d\dot{q}}\right) - \frac{d\text{T}'}{dq} = -\frac{d\text{V}}{dq}.$$

But this equation may be greatly simplified, for

$$u = \frac{d}{dt}\left(p_1 + \frac{\Sigma m_1 x_1}{\Sigma m_1}\right);$$

or, if we assume

$$\text{U} = \frac{\Sigma m_1 x_1}{\Sigma m_1},$$

we have

$$u = \dot{p}_1 + \frac{d\text{U}}{dt};$$

but U is a function of $q_1, \ldots q_{n-3}$, and does not contain t or $\dot{q}_1, \ldots \dot{q}_{n-3}$, or p_1, p_2, p_3, whence we may write

$$u = \dot{p}_1 + \Sigma \frac{d\text{U}}{dq}\dot{q}.$$

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It follows that

$$\frac{du}{d\dot{q}} = \frac{dU}{dq},$$

$$\frac{du}{dq} = \frac{d^2U}{dt\,dq};$$

and finally

$$\frac{d}{dt}\left(\frac{du}{d\dot{q}}\right) - \frac{du}{dq} = 0.$$

Similarly it may be shown that

$$\frac{d}{dt}\left(\frac{dv}{d\dot{q}}\right) - \frac{dv}{dq} = 0,$$

$$\frac{d}{dt}\left(\frac{dw}{d\dot{q}}\right) - \frac{dw}{dq} = 0;$$

and therefore the equation of motion reduces to

$$\frac{d}{dt}\left(\frac{dT'}{d\dot{q}}\right) - \frac{dT'}{dq} = -\frac{dV}{dq}.$$

By slightly altering the method of expressing the function T, the final result may be stated in the following theorem:—

Let V denote the potential energy of a system of attracting bodies.

Let T′ denote the sum of the kinetic energies of the rotations of each body of the system about its centre of inertia added to

$$\Sigma \frac{m_1 m_2}{2\Sigma m_1} [1, 2],$$

where the symbol [1, 2] denotes the square of the relative velocity of the centres of inertia of the masses m_1 and m_2, and the summation includes every pair of the bodies of the system.

Let $q_1, \ldots q_{n-3}$ be $n-3$ generalised coordinates adequate to specify everything about the position and aspect of the system at a given epoch, except its actual situation in space, then the $n-3$ equations which will determine the motion are

$$\frac{d}{dt}\left(\frac{dT'}{d\dot{q}_1}\right) - \frac{dT'}{dq_1} = -\frac{dV}{dq_1},$$

&c.

$$\frac{d}{dt}\left(\frac{dT'}{d\dot{q}_{n-3}}\right) - \frac{dT'}{dq_{n-3}} = -\frac{dV}{dq_{n-3}}.$$

In these equations the three objectionable coordinates do not appear, and thus the required elimination has been effected.